

§ 13. 测地线

这一节我们讨论内蕴曲面 (D, I) 上的曲线理论.

例 4.2: $(D, I) = (\mathbb{R}^2, d\tilde{x}^2 + d\tilde{y}^2)$ I_0

$$\begin{array}{ccc} (a, b) & \xrightarrow{r} & D \\ \downarrow & & \downarrow \\ t & \longrightarrow & (\tilde{x}(t), \tilde{y}(t)) \\ & & \parallel \\ & & r(t) \end{array} \quad \text{为平面曲线}$$

其曲率定义为

$$\begin{aligned} k(t) &= \frac{\tilde{x}(t)\ddot{y}(t) - \tilde{y}(t)\ddot{x}(t)}{(\dot{\tilde{x}}(t)^2 + \dot{\tilde{y}}(t)^2)^{3/2}} \\ &= \frac{II(t)}{I(t)} \end{aligned}$$

$I(t), II(t)$ 的计算如下: (参考 §5, P.P 52-53)

$$I(t) = r^* (d\tilde{x}^2 + d\tilde{y}^2) = dt \cdot (\tilde{x}(t), \tilde{y}(t)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{y}}(t) \end{pmatrix} (dt)^2$$

\parallel \parallel \parallel
 $\langle dr, dr \rangle_{I_0}$ \parallel \parallel \parallel
 $(\dot{\tilde{x}}(t)^2 + \dot{\tilde{y}}(t)^2)$ 诱导度量

$$\begin{aligned} II(t) &= -\langle dr, dn \rangle_{I_0} = \langle \ddot{r}(t), n(t) \rangle_{I_0} (dt)^2 \\ &= \langle d^2 r, n \rangle_{I_0} \parallel \parallel \end{aligned}$$

其中 $n(t)$ 为单位法向量.

设 (D, I) 为一内蕴曲面. 曲线 γ

$\gamma: (a, b) \rightarrow D$ 的第一基本型为

$$I(t) = \langle dr, dr \rangle_I = \gamma^*(I)$$

$$= \left[(\dot{x}(t), \dot{y}(t)) \begin{pmatrix} E(t) & F(t) \\ F(t) & G(t) \end{pmatrix} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \right] (dt)^2$$

那么如何定义 γ 的第二基本型呢?

正确的定义是

$$II(t) = -\langle dr, \nabla_I n \rangle_I = \langle \nabla_I dr, n \rangle_I$$

而不是: $-\langle dr, dn \rangle_I$.

$I(t)$

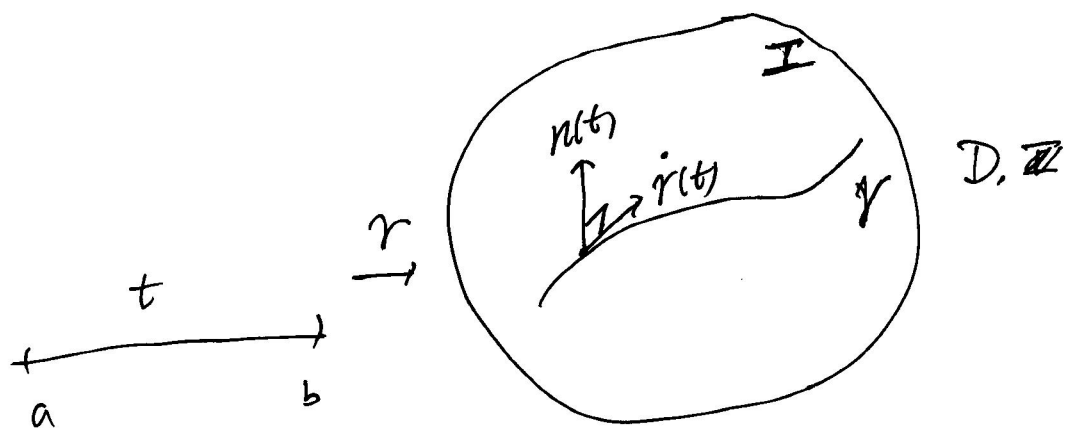
原因: n 是 D 上沿曲线 γ 的切向量. 对切向量切正确的求导

公式是 ∇_I , 而不是 d . !!!

定义: 曲线 γ 在 (D, I) 中的第二基本型为

$$II(t) = \langle \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t), n(t) \rangle_I (dt)^2 \text{ 其中}$$

$n(t)$ 为曲线 γ 的单位法向量



定义: 曲线 \$\gamma\$ 的测地曲率为

$$k_g(t) = \frac{\mathbb{I}(t)}{I(t)} = \frac{\langle \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t), n(t) \rangle_{\mathbb{I}}}{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\mathbb{I}}}$$

定义: 曲线 \$\gamma\$ 称为 \$(D, \mathbb{I})\$ 中测地线, 若 $k_g \equiv 0$.

$$\text{即 } \mathbb{I}(t) \equiv 0.$$

命题: 对曲线 \$\gamma(t)\$, 我们总可以重新参数化, 使得 \$\gamma(s)\$ 有性质,

$$\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\mathbb{I}} = 1, \text{ 即 } s \text{ 为 } \gamma \text{ 的弧长参数.}$$

证明: 令 $s(t_0) = \int_a^t \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\mathbb{I}}} dt$, $t_0 \in (a, b)$.

$$\text{则有 } \langle \dot{r}(s), \dot{r}(s) \rangle_{\mathbb{I}}$$

$$= \left\langle \dot{r}(t) \frac{dt}{ds}, \dot{r}(t) \frac{dt}{ds} \right\rangle_{\mathbb{I}}$$

$$= \left(\frac{dt}{ds} \right)^2 \langle \dot{r}(t), \dot{r}(t) \rangle_{\mathbb{I}}$$

$$= \left(\frac{dt}{ds} \right)^2 \cdot \left(\frac{d}{dt} \right)^2 = 1$$

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以下, 我们总是取 $r(s) = r(t)$, 其中 s 为弧长参数.

命题: 曲线 $r(s)$ 为测地线, 并且恰当

$r(s)$ 为自平行曲线, 即

$$\nabla_{\dot{r}(s)} \dot{r}(s) \equiv 0$$

证明:
$$K_g(s) = \frac{\langle \nabla_{\dot{r}(s)} \dot{r}(s), n(s) \rangle_{\mathbb{I}}}{\langle \dot{r}(s), \dot{r}(s) \rangle_{\mathbb{I}}}$$

$$= \langle \nabla_{\dot{r}(s)} \dot{r}(s), n(s) \rangle_{\mathbb{I}}$$

故, $\nabla_{\dot{r}(s)} \dot{r}(s) \equiv 0 \Rightarrow K_g(s) \equiv 0$

反之: 因为 $\langle \dot{r}(s), \dot{r}(s) \rangle_{\mathbb{I}} \equiv 1$

$$\Rightarrow d(\langle \dot{r}(s), \dot{r}(s) \rangle_{\mathbb{I}}) \equiv 0$$

$$\begin{aligned} \text{但 } d(\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\mathbb{I}}) \\ &= 2 \langle \nabla \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\mathbb{I}} + \langle \dot{\gamma}(s), \nabla \dot{\gamma}(s) \rangle_{\mathbb{I}} \\ &= 2 \langle \nabla \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\mathbb{I}} \equiv 0 \end{aligned}$$

$$\Rightarrow \frac{d \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\mathbb{I}}}{ds} = 2 \langle \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s), \dot{\gamma}(s) \rangle_{\mathbb{I}} \equiv 0$$

$$\Rightarrow \text{切向量场 } \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) \perp \dot{\gamma}(s)$$

$$\text{故 } \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) = \cancel{k_g(s) \dot{\gamma}(s)} \parallel n(s)$$

$$\text{即 } \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) = k_g(s) \cdot n(s)$$

$$\Rightarrow \text{若 } k_g(s) \equiv 0, \text{ 则 } \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) \equiv 0.$$

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自平行曲线 (弧长参数), 即该曲线的切向量场沿着该曲线平行.

推论: 给定 D 中一点 P , 及 P -个单位切向量 $v \in T_P D$. 存在且唯一一条

测地线 $\gamma(s)$, 使得 $P = \gamma(0)$, $v = \dot{\gamma}(0)$.

证明: 任取 e_1, e_2 为 D 上单正交标架. 则记

$$\nabla \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix} \wedge \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}; \quad \gamma(s) = (f_1(s), f_2(s))$$

求解常微分方程组

$$\begin{cases} \nabla_{\dot{r}(s)} \dot{r}(s) \equiv 0 \\ r(0) = P \\ \dot{r}(0) = v \end{cases}$$

记 $u^1 = u, u^2 = v, \dots, \frac{\partial}{\partial u} = \begin{pmatrix} \frac{\partial}{\partial u^1} \\ \frac{\partial}{\partial u^2} \end{pmatrix}, e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$

$$\begin{pmatrix} \frac{\partial}{\partial u^1} \\ \frac{\partial}{\partial u^2} \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \Leftrightarrow \frac{\partial}{\partial u} = A \cdot e$$

$$f = (f_1, f_2), \dot{f} = (\dot{f}_1, \dot{f}_2)$$

则有 $\nabla \dot{r}(s) = \nabla \left(\dot{f} \frac{\partial}{\partial u} \right)$

$$= \nabla (\dot{f} A e) = d(\dot{f} A) e + (\dot{f} A) w \cdot e$$

$$= (d(\dot{f} A) + (\dot{f} A) w) e$$

$$\Rightarrow \nabla_{\dot{r}(s)} \dot{r}(s) \equiv 0 \Leftrightarrow$$

$$(*) \quad \left(\dot{f} \frac{\partial}{\partial u} \right) (d(\dot{f} A) + (\dot{f} A) w) \equiv 0, \text{ 其中 } A, w \text{ 为已知}$$

(*) 是 f_1, f_2 的常微分方程组。

根据常微分方程理论, 存在满足初始条件的解存在且唯一。

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测地曲率在正交坐标系的计算公式:

命题: 设 $I = E du^2 + G dv^2$,

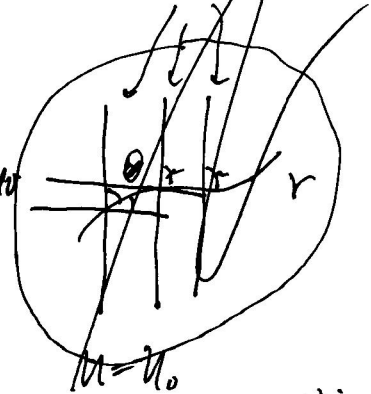
设 $\gamma = \gamma(s)$ 为 D 上 $-3m$ 长参数曲线. 则

$$k_g(s) = \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \log E}{\partial v} \cos \theta + \frac{1}{2\sqrt{E}} \frac{\partial \log G}{\partial u} \sin \theta,$$

$$= \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \log E}{\partial v} \cos \theta + \frac{1}{2\sqrt{E}} \frac{\partial \log G}{\partial u} \sin \theta,$$

其中 $\theta = \theta(s)$ 为曲线与 u -线的夹角. v -线

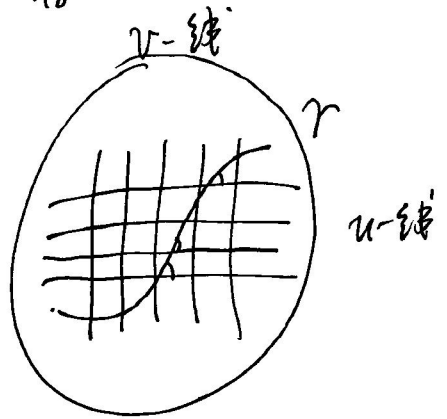
证明: $w_1 = \sqrt{E} du, w_2 = \sqrt{G} dv$
 $w_{12} = -\frac{(\sqrt{E})_v}{\sqrt{G}} du + \frac{(\sqrt{G})_u}{\sqrt{E}} dv$
 $e_1 = \frac{1}{\sqrt{E}} \frac{\partial}{\partial u}, e_2 = \frac{1}{\sqrt{G}} \frac{\partial}{\partial v}$



$$\dot{\gamma}(s) = \cos \theta \cdot e_1 + \sin \theta \cdot e_2$$

$$n(s) = -\sin \theta \cdot e_1 + \cos \theta \cdot e_2$$

$$\nabla(\dot{\gamma}(s)) =$$



$$k_g(s) = \langle \nabla_{\dot{\gamma}(s)} \dot{\gamma}(s), n(s) \rangle$$

$$= \left\langle -\sin \theta \cdot \frac{d\theta}{ds} \cdot e_1 + \cos \theta \frac{d\theta}{ds} e_2, -\sin \theta e_1 + \cos \theta e_2 \right\rangle$$

$$+ \left\langle \cos \theta \frac{dw_{12}}{ds} \cdot e_2 - \sin \theta \frac{dw_{12}}{ds} e_1, -\sin \theta e_1 + \cos \theta e_2 \right\rangle$$

$$= \frac{d\alpha}{ds} + \frac{dw_2}{ds}$$

$$\frac{dw_2}{ds} = - \frac{(\sqrt{E})_v}{\sqrt{G}} \cdot \frac{dy}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}} \cdot \frac{dv}{ds}$$

注意到

$$\cos\theta \cdot e_1 + \sin\theta \cdot e_2 = \dot{r}(s) = \dot{u}(s) \frac{\partial}{\partial u} + \dot{v}(s) \frac{\partial}{\partial v}$$

$$\Rightarrow \dot{u}(s) = \frac{\cos\theta}{\sqrt{E}} \quad ; \quad \dot{v}(s) = \frac{\sin\theta}{\sqrt{G}}$$

$$\Rightarrow k_g(s) = \frac{d\alpha}{ds} + \frac{(\sqrt{E})_v}{\sqrt{G}} \frac{\cos\theta}{\sqrt{E}} + \frac{(\sqrt{G})_u}{\sqrt{E}} \frac{\sin\theta}{\sqrt{G}}$$

$$= \frac{d\alpha}{ds} - \frac{\partial \log \sqrt{E}}{\partial v} \frac{1}{2\sqrt{G}} \cdot \cos\theta + \frac{\partial \log \sqrt{G}}{\partial u} \frac{1}{2\sqrt{E}} \sin\theta$$

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测地线与变分原理:

定理: $P, Q \in D$, 曲线 $\gamma: [0, 1] \rightarrow D$, 起点 $\gamma(0) = P$,
 终点 $\gamma(1) = Q$.

若 γ 是所有连接 P 与 Q 中最短的曲线, 则

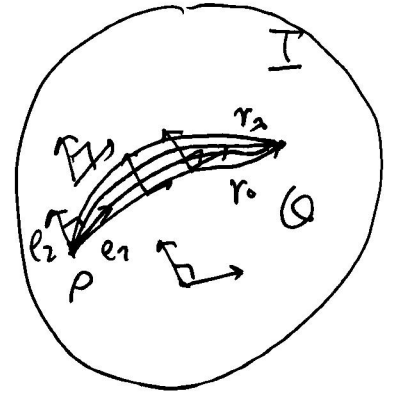
γ 是测地线. 即测地线是弧长变分的 Euler-Lagrange 方程.

证明: 不妨设

$\gamma_0 = \gamma_0(s) = (u_1(s), u_2(s))$ 为弧长参数曲线

取 D 上的正交标架 e_1, e_2 , 使得

$$e_1(s) = \dot{\gamma}(s), \quad e_2(s) = \nu(s).$$



设沿 γ_0 , 有

$$e_2(s) = a^1(s) \frac{\partial}{\partial u^1} + a^2(s) \frac{\partial}{\partial u^2}$$

设 $f(s)$ 为 $[0, 1]$ 上的一光滑函数, 满足

$$f(0) = f(1) = 0.$$

现考虑: 曲面上-族曲线:

$$\gamma_\lambda(s) = \gamma_0(s) + \lambda f(s) a^2(s), \quad \lambda \in (-\varepsilon, \varepsilon)$$

则有 (1) $\gamma_\lambda(0) = \gamma_0(0) = P$, $\gamma_\lambda(1) = \gamma_0(1) = Q, \forall \lambda$.

$$(2) \gamma_{\lambda=0} = \gamma_0$$

$$(3) \left. \frac{\partial \gamma_\lambda}{\partial \lambda} \right|_{\lambda=0} = f \cdot e_2$$

曲线族 $\{\gamma_\lambda\}$ 称为 γ_0 的一个扰动.

令 $L(\lambda)$ 为 r_λ 的弧长.

由假设 $L(\omega) = \inf_{\lambda \in (-\varepsilon, \varepsilon)} L(\lambda)$, 故有

$$\frac{dL(\lambda)}{d\lambda} \Big|_{\lambda=0} = 0.$$

故因为 $L(\lambda) = \int_0^1 \left\| \frac{\partial r_\lambda(s)}{\partial s} \right\| ds$

故 $\frac{dL(\lambda)}{d\lambda} \Big|_{\lambda=0} = \int_0^1 \left(\left\| \frac{\partial r_\lambda(s)}{\partial s} \right\|^{-\frac{1}{2}} \cdot \left\langle \frac{\partial}{\partial \lambda} \left(\frac{\partial r_\lambda}{\partial s} \right), \frac{\partial r_\lambda}{\partial s} \right\rangle \right) \Big|_{\lambda=0} ds$

$\left\| \frac{\partial r_\lambda(s)}{\partial s} \right\| \Big|_{\lambda=0} = 1$

$$= \int_0^1 \left\langle \frac{\partial}{\partial \lambda} \left(\frac{\partial r_\lambda}{\partial s} \right), \frac{\partial r_\lambda}{\partial s} \right\rangle \Big|_{\lambda=0} ds$$

$\int_0^1 \left\langle \frac{\partial}{\partial s} \left(\frac{\partial r_\lambda}{\partial \lambda} \right), \frac{\partial r_\lambda}{\partial s} \right\rangle \Big|_{\lambda=0} ds$

$\int_0^1 \left\langle \frac{\partial}{\partial s} (f \cdot e_2), e_1 \right\rangle ds$

$= - \int_0^1 \left\langle f \cdot e_2, \frac{d e_1}{ds} \right\rangle ds$

$= - \int_0^1 f \cdot k_g(s) ds \equiv 0, \quad \forall f$

$\Rightarrow k_g \equiv 0.$

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测地线在外蕴曲面上的表现：

设内蕴曲面 (D, I) , 由

$$\gamma: D \rightarrow (\mathbb{R}^3, dx^2 + dy^2 + dz^2)$$

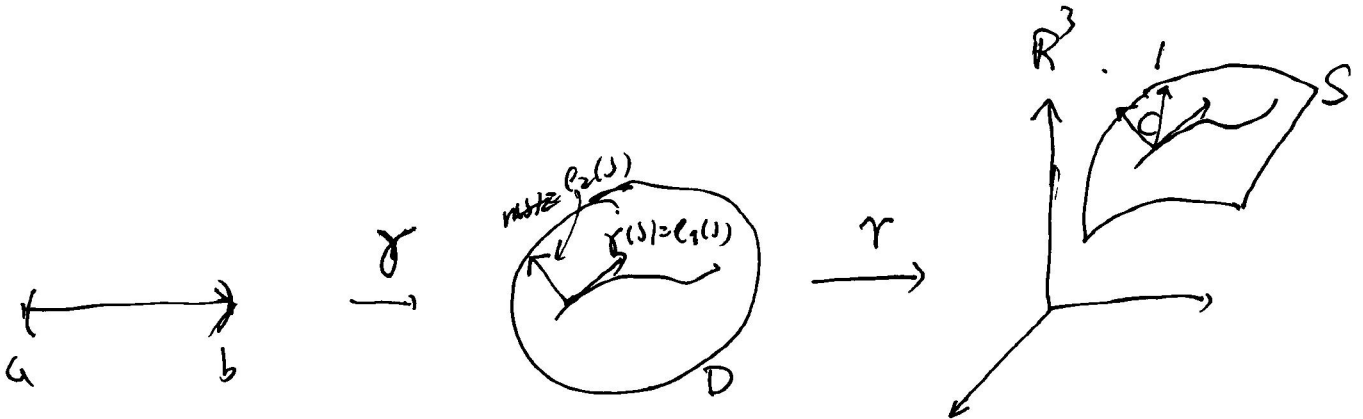
诱导。

则 (D, I) 上曲线 γ : (弧长参数)

$$\gamma: (a, b) \rightarrow D$$

可视为 \mathbb{R}^3 中空间曲线:

$$\tilde{\gamma}: (a, b) \xrightarrow{\gamma} D \xrightarrow{\gamma} \mathbb{R}^3$$



命题: 记 $K(s)$ 为 $\tilde{\gamma}$ 在 \mathbb{R}^3 中的曲率. 则有

$$K^2 = k_g^2 + k_n^2$$

其中 k_g 为 γ 的测地曲率, k_n 为 $\tilde{\gamma}$ 的法曲率.

证明:

$$k = |\ddot{\tilde{\gamma}}(s)|$$

而取 $(0, l)$ 上两正交标架, $\{e_1, e_2\}$, 使得

$$\dot{\tilde{\gamma}}(s), \quad e_1(s) = \dot{\tilde{\gamma}}(s)$$

$$e_2(s) = \frac{\nabla_{\dot{\tilde{\gamma}}(s)} \dot{\tilde{\gamma}}(s)}{|\nabla_{\dot{\tilde{\gamma}}(s)} \dot{\tilde{\gamma}}(s)|}$$

记 γ_* : 我们得 曲面 S 上单位正交标架:

$$\left\{ \begin{array}{ccc} e_1 & , & e_2 & , & e_3 \\ \parallel & & \parallel & & \parallel \\ \gamma_* e_1 & , & \gamma_* e_2 & , & e_3 \end{array} \right\}$$

$$\text{则 } \ddot{\tilde{\gamma}}(s) = \langle \ddot{\tilde{\gamma}}(s), e_2 \rangle e_2 + \langle \ddot{\tilde{\gamma}}(s), e_3 \rangle e_3$$

$$\text{根据定义: } k_n(s) = \langle \ddot{\tilde{\gamma}}(s), n \rangle = \langle \ddot{\tilde{\gamma}}(s), e_3 \rangle$$

$$k_g(s) = \langle \nabla_{\dot{\tilde{\gamma}}(s)} \dot{\tilde{\gamma}}(s), e_2(s) \rangle_I$$

$$= \langle \gamma_* (\nabla_{\dot{\tilde{\gamma}}(s)} \dot{\tilde{\gamma}}(s)), \gamma_* e_2(s) \rangle$$

$$= \langle Pr \cdot \ddot{\tilde{\gamma}}(s), \gamma_* e_2(s) \rangle$$

$$Pr: T\mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$k^2 = \langle \ddot{\tilde{\gamma}}(s), e_2(s) \rangle^2$$

$$\Rightarrow |\ddot{\tilde{\gamma}}(s)|^2 = k_g^2 + k_n^2$$

#

推论: $\sum S \subset \mathbb{R}^3$ 为曲面, CCS 为 \mathbb{R}^3 中直线, 则 C 是 S 上的测地线 168

测地线.

证明: 对于直线 $C \subset \mathbb{R}^3$, 我们有 $k \equiv 0$. 故

$$k_g^2 + k_n^2 = k^2 \equiv 0 \Rightarrow k_g \equiv 0$$

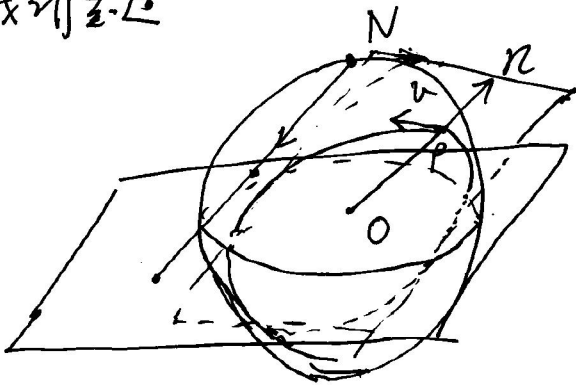
故 $C \subset S$ 为测地线.

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测地线的例子:

$$(1) \left(\mathbb{R}^2, \frac{du^2 + dv^2}{(1+u^2+v^2)^2} \right).$$

我们知道



$S^2 \setminus \{N\} \subset \mathbb{R}^3$ 在球极投影下

与 $(\mathbb{R}^2, \frac{du^2 + dv^2}{(1+u^2+v^2)^2})$ 等距同构.

故计算 $(\mathbb{R}^2, \frac{du^2 + dv^2}{(1+u^2+v^2)^2})$ 可以先确定 $S^2 \setminus \{N\}$ 在 \mathbb{R}^3 的测地线, 然后在球极投影下来即可.

对于任给 $P \in S^2 \setminus \{N\}$, $v \in T_P S^2$ 单位切向量.

我们考虑过 OP , 法向为 $u \times v$ 的平面截 S^2 得到

球面上大圆弧 C .

断言: C 的法方向与 $r(p)$ 同方向.
 \uparrow
 $p \in \Sigma$

事实上, 我们可以在点 p 处旋转, 使得 p 落在赤道上, 且 v 方向与赤道面平行. 即任给大圆弧 C 即为赤道.

但赤道的表达式为

$$r(\theta) = (\cos \theta, \sin \theta, 0), \quad \theta \neq 0 \text{ 为 } \sin \theta \text{ 的导数}$$

$$\text{故 } r'(\theta) = -(\sin \theta, \cos \theta, 0)$$

由此知 $\forall \theta$, 赤道大圆弧在 θ 处的法方向为

$$-n(\theta) = e_3$$

所以断言成立.

$$\text{故 } \begin{cases} \frac{d^2 r}{ds^2} = k_g e_2 + k_n \cdot n \\ \frac{d^2 r}{ds^2} \parallel n \end{cases} \Rightarrow k_g \equiv 0.$$

所以球面上任何曲线为大圆弧.

习题: 证明 $(\mathbb{R}^2, \frac{du^2 + dv^2}{(1+u^2+v^2)^2})$ 的任何曲线是测地线.

(2) Poincaré 上半平面..

$$\left(\begin{array}{c} H \\ \parallel \\ \{ (x,y) | y > 0 \} \end{array} , \quad \frac{dx^2 + dy^2}{y^2} \right)$$

$$w_1 = \frac{dx}{y}, \quad w_2 = \frac{dy}{y}, \quad w_{12} = \frac{dx}{y}$$

$$e_1 = y \frac{\partial}{\partial x}, \quad e_2 = y \frac{\partial}{\partial y}$$

设 $\gamma(s) = (x(s), y(s))$ 为测地线. 则

$$\dot{\gamma}(s) = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} = \underbrace{\left(\frac{\dot{x}}{y} \right)}_{\substack{\parallel \\ \mathfrak{g}^1}} e_1 + \underbrace{\left(\frac{\dot{y}}{y} \right)}_{\substack{\parallel \\ \mathfrak{g}^2}} e_2$$

$$\nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) = \left(\frac{d\mathfrak{g}^1}{ds} - \frac{d w_{12}}{ds} \mathfrak{g}^2 \right) e_1 + \left(\frac{d\mathfrak{g}^2}{ds} + \frac{d w_{12}}{ds} \mathfrak{g}^1 \right) e_2 \equiv 0$$

$$\Leftrightarrow \begin{cases} \frac{d\mathfrak{g}^1}{ds} - \mathfrak{g}^2 \frac{d w_{12}}{ds} = 0 \\ \frac{d\mathfrak{g}^2}{ds} + \mathfrak{g}^1 \frac{d w_{12}}{ds} = 0 \end{cases} \quad \text{由上式得} \quad \frac{d w_{12}}{ds} = \frac{\dot{x}}{y} = \mathfrak{g}^1$$

$$\Leftrightarrow \begin{cases} (\dot{\mathfrak{g}}^1) - \mathfrak{g}^2 \cdot \mathfrak{g}^1 = 0 \\ (\dot{\mathfrak{g}}^2) + \mathfrak{g}^1 \cdot \mathfrak{g}^1 = 0 \end{cases}$$

$$t_2 \neq 0 \Rightarrow$$

$$\frac{ds^1}{s^1} = s^2 ds = \frac{dy}{y}$$

$$\Rightarrow s^1 = cy$$

$$\text{设 } \alpha \quad (s^1)^2 + (s^2)^2 = 1, \quad (\text{令 } s^1 = \sin t) \text{ 有}$$

$$c^2 y^2 + \left(\frac{\dot{y}}{y}\right)^2 = 1$$

$$\Rightarrow ds = \frac{dy}{y\sqrt{1-c^2y^2}}$$

$$\text{令 } y = \frac{1}{c} \sin t, \quad \Rightarrow ds = \frac{dt}{\sin t}$$

$$\begin{aligned} \text{由 } s^2 &= \frac{\dot{y}}{y} = \frac{1}{y} \frac{dy}{ds} = \frac{c}{\sin t} \frac{dy}{dt} \frac{dt}{dy} = \frac{c}{\sin t} \frac{1}{c} \cos t \cdot \sin t \\ &= \cos t \end{aligned}$$

$$t_2 \quad s^1 = \sin t$$

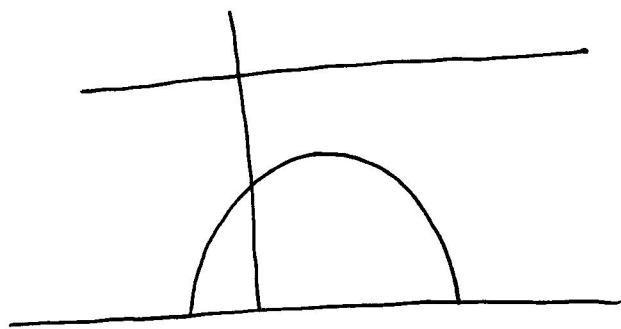
$$\begin{aligned} \Rightarrow x &= \int \tilde{x} ds = \int s^1 y ds = \int \sin t \frac{1}{c} \sin t \frac{ds}{dt} dt \\ &= -\frac{1}{c} \cos t + d \end{aligned}$$

$$\Rightarrow \begin{cases} x = -\frac{1}{c} \cos t + d \\ y = \frac{1}{c} \sin t \end{cases}$$

$$\Leftrightarrow (x-d)^2 + y^2 = \frac{1}{c^2}$$

另外: $\delta^1 \equiv 0$, 即 $x = a$ 的圆为经线.

因此, 经线为 x 轴上的半圆, 并与 x 轴平行的直线.



因此, 通过 Cayley 变换确定 Poincaré 圆盘

$$\left(\Delta, \frac{du^2 + dv^2}{1 - (u^2 + v^2)} \right) \text{ 上的测地线方程.}$$